Math 275D Lecture 3 Notes

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October 2, 2019

1 Uniform Continuity of Brownian Motion

1.1 Brownian Motion is Uniformly Continuous on $\mathbb{Z}[\frac{1}{2}]$

Recall that we wanted to define Brownian motion by defining it on $\mathbb{Z}[\frac{1}{2}]$, the dyadic rational numbers. We have $B(x) - B(y) \sim N(0, y-x)$ if $y, x \in \mathbb{Z}[\frac{1}{2}]$ and $B(0) = 0$. Since $(B(t_1), \dots, B(t_n))$ is a Gaussian vector, to show that $(B(2) - B(1)) \perp (B(3) - B(2))$, for example, we need only show that these are uncorrelated.

Lemma 1.1. *B on $\mathbb{Z}[\frac{1}{2}]$ is uniformly continuous.*

Here is the idea. Look at the interval $[0,1]$, and say $[a,b] \subseteq [0,1]$ is an interval of length ε. We know that $|B(b) - B(a)| \sim N(0, \varepsilon) \sim \sqrt{\varepsilon}$.

We need to prove that

$$\lim_{\varepsilon \downarrow 0} \sup_{\substack{|t-s| \leq \varepsilon \\ t,s \in \mathbb{Z}[\frac{1}{2}] \cap [0,1]}} |B(t) - B(s)| = 0.$$

The "bad event" where this does not happen is the union of small/basic bad events $\{|B(t) - B(s)| \geq \delta\}$, but we can show that the probability of the big bad event is $\ll \sum \mathbb{P}(\text{small bad event})$.

Proof. For each m and $\gamma > 0$, define $G_m^\gamma = \{\nexists k \text{ s.t.} |B(k/2^m) - B((k-1)/2^m)| \geq 2^{-\gamma m}\}$. Let $H^N = \bigcap_{m \geq N}^\infty G_m^\gamma$.

Now let's bound $|B(y) - B(x)|$. Find an interval $I_1 = [a,b] \subseteq [x,y]$ of length 2^{-N}. If we are in H^N, then $|B(b) - B(a)| \leq 2^{-\gamma N}$. Now let $I_2 = [a - 1/2^{N+1}, a]$ and $I_3 = [b, b + 1/2^{N+1}]$ if these are contained in $[x,y]$ and let them be $\varnothing$ otherwise. Then if we are in H^N, $|B(a) - B(a - 1/2^{N+1})| \leq 2^{-\gamma(N+1)}$ and similarly for I_3. Proceeding like this, we can split the interval $[x,y]$ into intervals I_k of length 2^{-n} for $n \geq N$ with at most 2 of each kind. So $[x,y] = \bigcup_{n=1}^\infty I_n$, and we get

$$|B(y) - B(x)| \leq 2 \sum_{m \geq N}^\infty 2^{m\gamma}.$$

So in H^N, if $|x-y| \leq 2^{-(N-1)}$, then $|B(y)-B(x)| \leq C2^{-N\gamma}$ for some constant $C > 0$.

To get uniform continuity, we don't want to have fixed H^N. Using Markov's inequality with the fact that $\mathbb{E}[X]^4] = 1/2^m$ when $X \sim N(0, 1/2^m)$,

$$1 - \mathbb{P}(G_m^\gamma) = 2^m \mathbb{P}(|B(1/2^m) - B(0)| \geq 2^{-\gamma m}) \leq \frac{2^{-2m}}{2^{-4\gamma m}} \leq 2^{(-1+4\gamma)m}.$$

So $1 - \mathbb{P}(H^N)$ is bounded by a convergent geometric series. We will finish the proof next time. $\square$